When vectors shoot at the Centres of a Triangle ….

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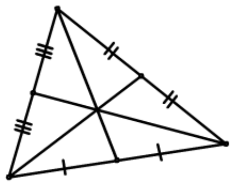
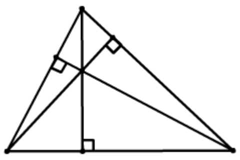
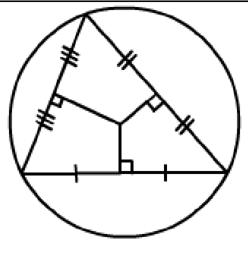
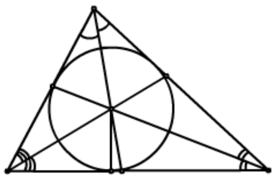
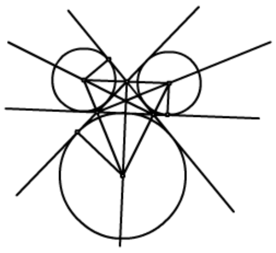
1. Introduction

				
Centroid(*G*) (重心)	**Orthocentre (*H*)** (垂心)	**Circumcentre (*O*)** (外接圓心)	**In-centre (*I*)** (內接圓心)	**Ex-centres (*E₁*, *E₂*, *E₃*)** (外心)

This article aims to find the vector forms of five classical centres of a triangle. It is assumed that the reader has learned a basic course of vector algebra. We use $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ to denote the position vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ of the three vertices of the triangle. Similarly, we use $\boldsymbol{X}$ to denote the position vector $\overrightarrow{OX}$ of any point X. We use unbold italic A, B, C to denote the angles of the triangle ABC and a, b, c the length of the sides opposite to the angles A, B, C respectively.

2. Centroid (*G*) (Refer to Figure 1)

Since BD : DC = 1 : 1, we have $\boldsymbol{D}=\dfrac{\boldsymbol{B}+\boldsymbol{C}}{2}$

Since AG : GD = 2 : 1, we have

$$\boldsymbol{G}=\frac{\boldsymbol{A}+2\boldsymbol{D}}{3}=\frac{\boldsymbol{A}+2\dfrac{\boldsymbol{B}+\boldsymbol{C}}{2}}{3}=\underline{\underline{\frac{\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}}{3}}} \qquad \text{…. (1)}$$

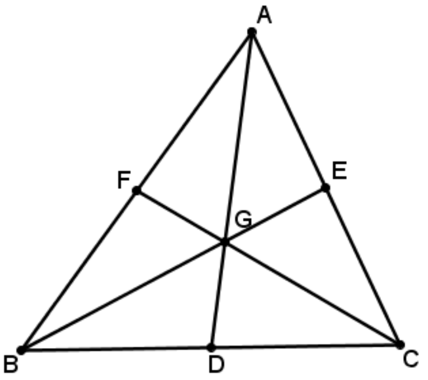


Figure 1

3. Some preliminary results : (Refer to Figure 2, 3 below)

(a) $\overrightarrow{OH}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$, where Orthocentre (H), Circum-centre (O)

Proof

We get around the problem a bit. We suppose $\overrightarrow{OH}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$ and then show that H is the orthocenter.

Since O is the circum-centre, $|\boldsymbol{OA}|=|\boldsymbol{OB}|=|\boldsymbol{OC}|$ …. (2)

$$\overrightarrow{AH}=\overrightarrow{OH}-\overrightarrow{OA}=\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right)-\overrightarrow{OA}=\overrightarrow{OB}+\overrightarrow{OC} \qquad \text{…. (3)}$$

and $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}$

$\therefore$ $\overrightarrow{AH}\bullet\overrightarrow{BC}=\left(\overrightarrow{OB}+\overrightarrow{OC}\right)\bullet\left(\overrightarrow{OC}-\overrightarrow{OB}\right)=\left|\overrightarrow{OC}\right|^2-\left|\overrightarrow{OB}\right|^2=0$, , by (2)

$\therefore$ $\overrightarrow{AH}\perp\overrightarrow{BC}$, similarly $\overrightarrow{BH}\perp\overrightarrow{CA}$, $\overrightarrow{CH}\perp\overrightarrow{AB}$

$\therefore$ H is the orthocenter.

If we can take the circum-centre (O) as origin and we simply write $\boldsymbol{H}=\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}$

(b) $\overrightarrow{AH}\tan A+\overrightarrow{BH}\tan B+\overrightarrow{CH}\tan C=0$

Let R be the radius of the circum-circle. By (3),

$$\left|\overrightarrow{AH}\right|^2=\left|\overrightarrow{OB}+\overrightarrow{OC}\right|^2=\left(\overrightarrow{OB}+\overrightarrow{OC}\right)\bullet\left(\overrightarrow{OB}+\overrightarrow{OC}\right)=\left|\overrightarrow{OB}\right|^2+\left|\overrightarrow{OC}\right|^2-2\overrightarrow{OB}\bullet\overrightarrow{OC}$$

$$=\left|\overrightarrow{OB}\right|^2+\left|\overrightarrow{OC}\right|^2-2\left|\overrightarrow{OB}\right|\left|\overrightarrow{OC}\right|\cos BOC\ =R^2+R^2-2R^2\cos 2A,$$ ∠ at centre twice ∠ at circumference

$\therefore \quad \left|\overrightarrow{AH}\right|^2 = 2R^2(1-\cos 2A) = 2R^2(2\cos^2 A) = (2R\cos A)^2 \qquad \therefore \quad \left|\overrightarrow{AH}\right| = \pm 2R\cos A$

For simplicity, we take ∠A, ∠B, ∠C to be acute angles. Same result can be got with obtuse angle. (note 1)

$$\therefore \quad \left|\overrightarrow{AH}\right| = +2R\cos A, \quad \left|\overrightarrow{BH}\right| = +2R\cos B, \quad \left|\overrightarrow{CH}\right| = +2R\cos C \qquad \text{.... (4)}$$

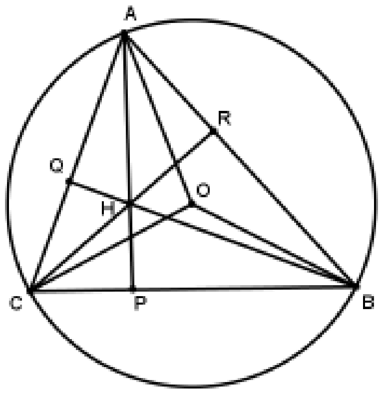


Figure 2

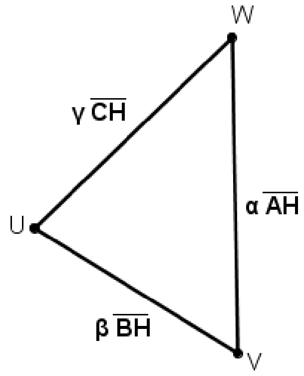


Figure 3

Since $\overrightarrow{AH}, \ \overrightarrow{BH}, \ \overrightarrow{CH}$ are non-parallel, we can let

$\alpha\overrightarrow{AH} + \beta\overrightarrow{BH} + \gamma\overrightarrow{CH} = 0$, where α, β, γ are constants.

We then form a triangle with sides :

$\overrightarrow{WV} = \alpha\overrightarrow{AH}, \quad \overrightarrow{VU} = \beta\overrightarrow{BH}, \quad \overrightarrow{UW} = \gamma\overrightarrow{CH}$

Note that $\overrightarrow{WV} \, // \, \overrightarrow{AH}, \quad \overrightarrow{VU} \, // \, \overrightarrow{BH}, \quad \overrightarrow{UW} \, // \, \overrightarrow{CH}$

By comparing the right diagrams of Figure 2 and Figure 3 ,

∠U = ∠BHR = 180° – ∠($\overrightarrow{BH}, \overrightarrow{CH}$) = 180° – ∠QHR = ∠A

Similarly, ∠V = ∠B, ∠W = ∠C

By sine law on ΔUVW, $\dfrac{\alpha\left|\overrightarrow{AH}\right|}{\sin U} = \dfrac{\beta\left|\overrightarrow{BH}\right|}{\sin V} = \dfrac{\gamma\left|\overrightarrow{CH}\right|}{\sin W} = 2R_1$,

where R_1 is the radius of circum-circle enclosing ΔUVW.

By (4), $\dfrac{\alpha(2R\cos A)}{\sin A} = \dfrac{\beta(2R\cos B)}{\sin B} = \dfrac{\gamma(2R\cos C)}{\sin C} = 2R_1$

$\alpha = \dfrac{R_1}{R}\tan A, \quad \beta = \dfrac{R_1}{R}\tan B, \quad \gamma = \dfrac{R_1}{R}\tan C$ and $\alpha\overrightarrow{AH} + \beta\overrightarrow{BH} + \gamma\overrightarrow{CH} = 0$

$$\Rightarrow \overrightarrow{AH}\tan A + \overrightarrow{BH}\tan B + \overrightarrow{CH}\tan C = 0 \qquad \text{.... (5)}$$

4. Orthocentre (*H*)

Since from (5), $\left(\overrightarrow{OH} - \overrightarrow{OA}\right)\tan A + \left(\overrightarrow{OH} - \overrightarrow{OB}\right)\tan B + \left(\overrightarrow{OH} - \overrightarrow{OC}\right)\tan C = 0$

$$\text{Solving,} \quad \overrightarrow{OH} = \frac{\overrightarrow{OA}\tan A + \overrightarrow{OB}\tan B + \overrightarrow{OC}\tan C}{\tan A + \tan B + \tan C} \quad \text{or} \quad \boldsymbol{H} = \frac{\boldsymbol{A}\tan A + \boldsymbol{B}\tan B + \boldsymbol{C}\tan C}{\tan A + \tan B + \tan C} \qquad \text{.... (6)}$$

$$\text{Reader may check similar result,} \quad \boldsymbol{H} = \frac{\boldsymbol{A}(a\sec A) + \boldsymbol{B}(b\sec B) + \boldsymbol{C}(c\sec C)}{a\sec A + b\sec B + c\sec C} \qquad \text{.... (7)}$$

The above results (6) and (7) are good even if O is not the cirum-centre, but any assigned origin.

5. Circum-centre (*O*) (Refer to Figure 2, 4)

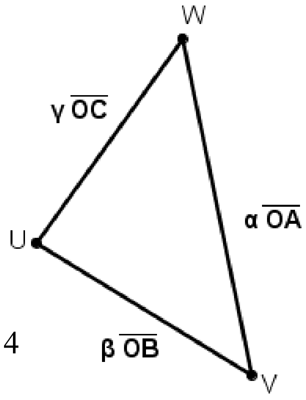


Figure 4

We still take O to be the circum-centre but with respect to any origin X.

We like to show that $\overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C = 0$ first .

We use the same technique as before and take $\alpha\overrightarrow{OA} + \beta\overrightarrow{OB} + \gamma\overrightarrow{OC} = 0$.

We can form a triangle UVW with sides $\overrightarrow{WV} = \alpha\overrightarrow{OA}, \quad \overrightarrow{VU} = \beta\overrightarrow{OB}, \quad \overrightarrow{UW} = \gamma\overrightarrow{OC}$

∠U = 180° – ∠($\overrightarrow{OB}, \overrightarrow{OC}$) = 180° – ∠BOC = 180° – 2 ∠A , ∠ at centre twice ∠ at circumference.

Similarly, ∠V = 180° – 2 ∠B , ∠W = 180° – 2 ∠C

By sine law on ΔUVW,

$\dfrac{\alpha\left|\overrightarrow{OA}\right|}{\sin U} = \dfrac{\beta\left|\overrightarrow{OB}\right|}{\sin V} = \dfrac{\gamma\left|\overrightarrow{OC}\right|}{\sin W} = 2R_1$, where R_1 is the radius of circum-circle enclosing ΔUVW.

$$\frac{\alpha R}{\sin 2A}=\frac{\beta R}{\sin 2B}=\frac{\gamma R}{\sin 2C}=2R_1 \qquad \Rightarrow \alpha=\frac{2R}{R_1}\sin 2A,\ \ \beta=\frac{2R}{R_1}\sin 2B,\ \ \gamma=\frac{2R}{R_1}\sin 2C$$

Since $\alpha\overrightarrow{OA}+\beta\overrightarrow{OB}+\gamma\overrightarrow{OC}=0$, therefore $\overrightarrow{OA}\sin 2A+\overrightarrow{OB}\sin 2B+\overrightarrow{OC}\sin 2C=0$

Now, $\left(\overrightarrow{XA}-\overrightarrow{XO}\right)\sin 2A+\left(\overrightarrow{XB}-\overrightarrow{XO}\right)\sin 2B+\left(\overrightarrow{XC}-\overrightarrow{XO}\right)\sin 2C=0$

$$\overrightarrow{XO}=\frac{\overrightarrow{XA}\sin 2A+\overrightarrow{XB}\sin 2B+\overrightarrow{XC}\sin 2C}{\sin 2A+\sin 2B+\sin 2C} \quad \text{or simply} \quad \boldsymbol{O}=\frac{\boldsymbol{A}\sin 2A+\boldsymbol{B}\sin 2B+\boldsymbol{C}\sin 2C}{\sin 2A+\sin 2B+\sin 2C} \qquad \text{.... (8)}$$

The reader may check similar result: $\boldsymbol{O}=\dfrac{\boldsymbol{A}(a\cos A)+\boldsymbol{B}(b\cos B)+\boldsymbol{C}(c\cos C)}{a\cos A+b\cos B+c\cos C}$ .... (9)

6. In-centre (*I*) (Refer to Figure 5 , 6)

We like to show that $a\overrightarrow{IA}+b\overrightarrow{IB}+c\overrightarrow{IC}=0$ first . We use the same technique as before and take

$\alpha\overrightarrow{IA}+\beta\overrightarrow{IB}+\gamma\overrightarrow{IC}=0$.

We can form a triangle UVW with sides

$\overrightarrow{WV}=\alpha\overrightarrow{IA},\quad \overrightarrow{VU}=\beta\overrightarrow{IB},\quad \overrightarrow{UW}=\gamma\overrightarrow{IC}$.

$\angle U = 180^\circ-\angle(\overrightarrow{IB},\overrightarrow{IC}) = t+u$

$\angle V = 180^\circ-\angle(\overrightarrow{IA},\overrightarrow{IB}) = u+s$

$\angle W = 180^\circ-\angle(\overrightarrow{IC},\overrightarrow{IA}) = s+t$

By sine law on ΔUVW, $\dfrac{\alpha\left|\overrightarrow{IA}\right|}{\sin U}=\dfrac{\beta\left|\overrightarrow{IB}\right|}{\sin V}=\dfrac{\gamma\left|\overrightarrow{IC}\right|}{\sin W}=2R_1$,

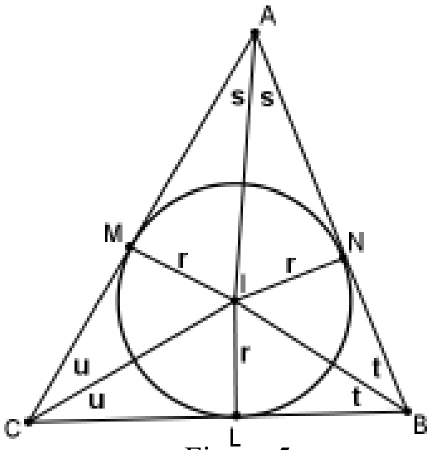


Figure 5

where R_1 is the radius of circum-circle enclosing ΔUVW.

$$\frac{\alpha\frac{r}{\sin s}}{\sin(t+u)}=\frac{\beta\frac{r}{\sin t}}{\sin(u+s)}=\frac{\gamma\frac{r}{\sin u}}{\sin(s+t)}=2R_1$$, where r is the radius of the in-circle.

$$\therefore\quad \alpha=\frac{2R_1}{r}\sin s\sin(t+u)=\frac{2R_1}{r}\sin s\sin\left(90^\circ-s\right)=\frac{2R_1}{r}\sin s\cos s=\frac{R_1}{r}\sin 2s$$

Similarly, $\beta=\dfrac{R_1}{r}\sin 2t,\ \ \gamma=\dfrac{R_1}{r}\sin 2u$.

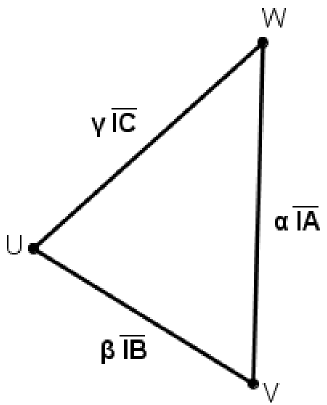


Figure 6

$$\alpha\overrightarrow{IA}+\beta\overrightarrow{IB}+\gamma\overrightarrow{IC}=0\Rightarrow(\sin 2s)\overrightarrow{IA}+(\sin 2t)\overrightarrow{IB}+(\sin 2u)\overrightarrow{IC}=0$$

$$\Rightarrow a\left(\frac{\sin 2s}{a}\right)\overrightarrow{IA}+b\left(\frac{\sin 2t}{b}\right)\overrightarrow{IB}+c\left(\frac{\sin 2u}{c}\right)\overrightarrow{IC}=0$$

$$\Rightarrow a\overrightarrow{IA}+b\overrightarrow{IB}+c\overrightarrow{IC}=0, \qquad \because\left(\frac{\sin 2s}{a}\right)=\left(\frac{\sin 2t}{b}\right)=\left(\frac{\sin 2u}{c}\right),$$ by sine law on ΔABC.

$$\Rightarrow a\left(\overrightarrow{OA}-\overrightarrow{OI}\right)+b\left(\overrightarrow{OB}-\overrightarrow{OI}\right)+c\left(\overrightarrow{OB}-\overrightarrow{OI}\right)=0,$$ O is any origin .

$$\Rightarrow\overrightarrow{OI}=\frac{a\overrightarrow{OA}+b\overrightarrow{OB}+c\overrightarrow{OC}}{a+b+c} \quad \text{or simply} \quad \boldsymbol{I}=\frac{a\boldsymbol{A}+b\boldsymbol{B}+c\boldsymbol{C}}{a+b+c} \qquad \text{.... (10)}$$

7. Ex-centres (E_1 , E_2 , E_3)

With similar method of proofs, the three ex-centres, formed by angle bisectors of exterior angles, can be found:

$$\boldsymbol{E_1}=\frac{-a\boldsymbol{A}+b\boldsymbol{B}+c\boldsymbol{C}}{-a+b+c},\ \ \boldsymbol{E_2}=\frac{a\boldsymbol{A}-b\boldsymbol{B}+c\boldsymbol{C}}{a-b+c},\ \ \boldsymbol{E_3}=\frac{a\boldsymbol{A}+b\boldsymbol{B}-c\boldsymbol{C}}{a+b-c} \qquad \text{.... (11)}$$

Note :

(1) For more detailed results and discussions, please refer to my site: http://www.qc.edu.hk/math/index.htm .

(2) The author would like to thank Ms. Chan Lap-lin, Head of the Department of Mathematics at Queen's College, for her careful proof-reading,